SOME IMPORTANT INEQUALITY

We have learned from previous files to transform algebraic expressions. With suitable transformations we can achieve to have conclusion on the sign of these expressions, that is, to claim with certainty that the expression is positive or negative for all values of variables that occur in it.

1) $x^2 \geq 0$ for all $x \in R$ Square of an expression is always positive or zero (for x = 0)

Examples: $\rightarrow x^2+4x+4=(x+2)^2 \geq 0$ for $\forall x \in R$

$\rightarrow -a^2+2a-1=-(a-1)^2 \leq 0$ for $\forall a \in R$

$\rightarrow x^2-xy+y^2 \geq 0$ because

$$x^2-xy+\left(\frac{y}{2}\right)^2-\left(\frac{y}{2}\right)^2+y^2=\left(x-\frac{y}{2}\right)^2-\frac{y^2}{4}+y^2=\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{4}$$

$\left(x-\frac{y}{2}\right)^2 \geq 0$ and $\frac{3y^2}{4} \geq 0$,

2) $\frac{x^2+y^2+z^2+3}{2} \geq x+y+z$

Proof:

$$\begin{aligned}(x-1)^2 &\geq 0\\ (y-1)^2 &\geq 0\\ (z-1)^2 &\geq 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}&(x-1)^2+(y-1)^2+(z-1)^2 \geq 0\\ &x^2-2x+1+y^2-2y+1+z^2-2z+1 \geq 0\\ &x^2+y^2+z^2+3 \geq 2x+2y+2z\\ &x^2+y^2+z^2+3 \geq 2(x+y+z)\\ &\frac{x^2+y^2+z^2+3}{2} \geq x+y+z\end{aligned}$$

3) **Demonstrate that** $\forall a>0 \Rightarrow a+\frac{1}{2} \geq 2$

Proof:

$$(a-1)^2 \geq 0$$
$$a^2-2a+1 \geq 0$$
$$a^2+1 \geq 2a \,/:a$$
$$a+\frac{1}{a} \geq 2$$

4) **Prove that for** $\forall x \geq 0$ **and** $\forall y \geq 0$ **is** $\sqrt{xy} \leq \frac{x+y}{2}$ **(geometric middle** $\leq$ **arithmeticalmiddle)**

Proof:

$$\left(\sqrt{x}+\sqrt{y}\right)^2 \geq 0$$
$$\sqrt{x}^2 - 2\sqrt{x}\sqrt{y} + \sqrt{y}^2 \geq 0$$
$$x - 2\sqrt{xy} + y \geq 0$$
$$x + y = 2\sqrt{xy} \,/ : 2$$
$$\frac{x+y}{2} \geq \sqrt{xy}$$

Of course, equality is if x = y.

5) Demonstrate that: $\forall x, y, z \quad (0 \leq x, 0 \leq y, 0 \leq z) \quad \Rightarrow \quad \sqrt[3]{xyz} \leq \frac{a^3+b^3+c^3}{3}$

Proof: First to replace
$$x = a^3$$
$$y = b^3$$
$$z = c^3$$

$$\sqrt[3]{xyz} \leq \frac{a^3+b^3+c^3}{3}$$
$$\sqrt[3]{a^3b^3c^3} \leq \frac{a^3+b^3+c^3}{3}$$
$$abc \leq \frac{a^3+b^3+c^3}{3}$$
$$3abc \leq a^3+b^3+c^3$$
$$a^3+b^3+c^3-3abc \geq 0$$

How is : $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac)$

From here we have $a+b+c \geq 0$ certainly, because $0 \leq x, 0 \leq y, 0 \leq z$

and $$a^2+b^2+c^2-ab-bc-ac = \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 0$$

So the product of two such expression is > 0 and is: $abc \leq \frac{a^3+b^3+c^3}{3}$ So: $\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$

Sign = is if x = y = z